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COMPACT MANIFOLDS OF NONPOSITIVE CURVATURE

H. BLAINE LAWSON, JR. & SHING TUNG YAU

0. Introduction and statement of results

Let M be a compact C^∞ riemannian manifold of nonpositive curvature[1] and with fundamental group π. It is well known [8, p. 102] that M is a $K(\pi, 1)$ and thus completely determined up to homotopy type by π. In light of this fact it is natural to ask to what extent the riemannian structure of M is determined by the structure of π, and the intent of this paper is to demonstrate that rather strong implications of this sort exist.

In the case that M has strictly negative curvature, the group π is known to be highly noncommutative. Every abelian, in fact, every solvable, subgroup of π is cyclic [3]. It is therefore a plausible conjecture that in the nonpositive curvature case, π will possess large amounts of commutativity only under special geometric circumstances. We shall show that this is true, that indeed those properties of π which involve commutativity have a dramatic reflection in the riemannian structure of M.

Our first theorem concerns abelian subgroups of π, which, since no element of π has finite order [8, p. 103], must be torsion free. As remarked above, when M is negatively curved, every abelian subgroup has rank one. However, when the curvature of M is simply nonpositive, we prove the following.

The flat torus theorem. *There exists an abelian subgroup of rank k in π if and only if there exists a flat k-torus isometrically and totally geodesically immersed in M.*

The second theorem concerns the case where π is a product of groups. In particular, we shall prove:

The splitting theorem. *Let M be real analytic and assume that π has no center. If π can be expressed as a direct product of groups $\pi = \mathscr{A}_1 \times \cdots \times \mathscr{A}_N$, then M is isometric to a riemannian product $M = M_1 \times \cdots \times M_N$, where $\pi_1(M_k) = \mathscr{A}_k$ for $k = 1, \cdots, N$.*

It is shown in § 4 that in the case that π has a nontrivial center, the splitting theorem, as stated, is not true. However, by a slight weakening of the conclusion, one can obtain a similar theorem for the general case.

As one may by now suspect, the appearance of a nontrivial center in π must

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[1] Throughout the paper curvature refers to sectional curvature.

also have strong geometric consequences. In fact, from the work of J. A. Wolf in [12] one has

The center theorem. *Let $\mathscr{Z}$ be the center of π. Then $\mathscr{Z} \simeq k\boldsymbol{Z}$ for some $k \geq 0$, and there exists a foliation of M by totally geodesic, flat k-tori. Furthermore, there exists an abelian covering $T^k \times M' \to M$ of M by a riemannian product of a flat torus and another manifold M'. Let $\mathscr{N} = \pi_1(M')$ and let $\mathscr{A}$ be the abelian covering group. Then $\mathscr{N}$ is a normal subgroup of π which contains $[\pi, \pi]$, and the following sequences are exact:*

$$1 \to \mathscr{Z} \times \mathscr{N} \to \pi \to \mathscr{A} \to 1 ,$$

$$0 \to \mathscr{Z} \times (\mathscr{N}/[\pi, \pi]) \to H_1(M, \boldsymbol{Z}) \to \mathscr{A} \to 0 .$$

As particular consequences of this theorem we have that if $\mathscr{Z} \simeq k\boldsymbol{Z}$, then:

(a) there exist k linearly independent globally parallel vector fields on M,

(b) the torus group T^k acts effectively by isometries on M.

In § 5 we show that these geometric quantities completely characterize the center of π, namely:

(a′) *Suppose there exist exactly k linearly independent globally parallel vector fields on M. Then* rank $(\mathscr{Z}) = k$.

(b′) *Let $I(M)$ be the group of isometries of M. Then $I(M)^0 \simeq T^k$ where $k =$* rank $(\mathscr{Z})$. *Furthermore, if $g \in I(M) \sim I(M)^0$, then g is not homotopic to the identity.*

Part (b′) together with the center theorem gives a generalization of a theorem of T. Frankel to manifolds of nonpositive curvature (§ 6).

In all of the above theorems the compactness of M is required. In fact, Bishop and O'Neill have shown that there exists a complete metric of constant negative curvature on $\boldsymbol{R} \times F$ where F is any compact manifold which admits a flat riemannian metric (e.g., a torus) [2, Cor. 7.10].

However, in the last section we show that certain of the above results can be shown to hold for complete nonpositively curved manifolds of finite volume. In particular, a form of Gottlieb's theorem is established for such cases.

We are indebted to S. Kobayashi for several helpful suggestions.

Note. Since writing this paper we have learned that J. A. Wolf and D. Gromoll[2] have obtained independent and somewhat different proofs of the first two theorems, including a C^∞ version of the splitting theorem.

1. Definitions, notation and basic lemmas

Thoughout the proofs of the main theorems of this paper we will need to make repeated use of certain established facts concerning manifolds of non-

[2] **Added in proof.** D. Gromoll & J. A. Wolf, *Some relations between the metric structure and the algebraic structure of the fundamental group in manifolds of nonpositive curvature,* Bull. Amer. Math. Soc. **77** (1971) 545–552.

positive curvature. Therefore, for the sake of completeness and clarity we begin with a brief discussion of these facts.

We shall always denote by M a compact riemannian manifold of nonpositive curvature, the metric on M by $\langle \cdot, \cdot \rangle$, the riemannian connection by ∇, and the riemannian curvature by $R_{X,Y} = \nabla_{[X,Y]} - \nabla_X \nabla_Y + \nabla_Y \nabla_X$. For any smooth curve $\gamma(t)$ in M, we denote the velocity vector field of γ by $\vec{\gamma}(t)$ or $(d\gamma/dt)(t)$.

We begin by stating some well known properties of compact manifolds of nonpositive curvature [1].

Fact 1. *For all $p \in M$ the exponential map* $\exp_p \colon T_p(M) \to M$ *is a covering map (the universal covering map).*

We shall always choose our universal coverings this way, where we assume $T_p(M)$ to have the lifted metric and $0 \in T_p(M)$ to be the distinguished point above p. This universal covering shall always be denoted $\pi \colon \tilde{M} \to M$.

Fact 2. *If $\bar{M} \subset M$ is a compact totally geodesic submanifold, then* $\exp \colon N(\bar{M}) \to M$ *is a covering map from the normal bundle of $\bar{M}$ to M. If $\bar{M} \subset \tilde{M}$ is a complete totally geodesic submanifold, then* $\exp \colon N(\bar{M}) \to \tilde{M}$ *is a diffeomorphism* [6].

Fact 3. *For any $a \in \pi_1(M, p)$ there is a unique geodsic loop at p which represents a. This is the loop of shortest length in a, and corresponds to the unique geodesic in $\tilde{M}$ from 0 to $A(0)$ where A is the deck transformation corresponding to a.*

Fact 4. *Consider the diffeomorphism* $\exp_{\bar{p}} \colon T_{\bar{p}}(\tilde{M}) \to \tilde{M}$ *where $T_{\bar{p}}(\tilde{M})$ is assumed to have the usual flat metric induced from the inner product. Then* $\exp_{\bar{p}}$ *does not decrease the length of curves. In particular, it does not decrease distance.*

We now examine a method for constructing totally geodesic submanifolds in $\tilde{M}$. The construction is based on the following notion. A subset $C \subset \tilde{M}$ is said to be *completely convex* if for all points $p, q \in C, p \neq q$, the unique infinite (i.e., maximal) geodesic $\gamma_{p,q}$ passing through p and q is contained in C. It follows from the work of Cheeger and Gromoll [4] that any completely convex subset of $\tilde{M}$ is a complete totally geodesic submanifold.

Let γ_1 and γ_2 be infinite geodesics in $\tilde{M}$. Then γ_1 and γ_2 are said to be of the *same type* if

$$\sup_s [\text{dist}(\gamma_1(s), \gamma_2)] + \sup_s [\text{dist}(\gamma_2(s), \gamma_1)] < \infty .$$

This defines an equivalence relation on the infinite geodesics in $\tilde{M}$. Using Fact 4 and the classical Gauss-Bonnet Theorem, J. A. Wolf [12] established the following:

Proposition 1. *Suppose γ_1 and γ_2 are distinct infinite geodesics of the same type in $\tilde{M}$. Then there exists a flat totally geodesic ribbon of surface of uniform width between γ_1 and γ_2.*

It follows that for any infinite geodesic γ in $\tilde{M}$ the set

$$C_\gamma = \cup\{\gamma' \in G : \gamma' \text{ is the same type as } \gamma\} ,$$

where G denotes the collection of infinite geodesics in $\tilde{M}$, is geodesically convex. Furthermore, if the manifold $\tilde{M}$ is real analytic, then C_γ is completely convex. Therefore we have

Corollary 1. *Suppose that the manifold $\tilde{M}$ is real analytic. Then for any infinite geodesic γ, the set C_γ is a complete totally geodesic submanifold.*

We now recall the notion of a *fundamental vector field* on $\tilde{M}$ introduced in [12]. Let $p \in M$, and choose the universal covering $\pi: \tilde{M} \to M$ (with natural base points) as in Fact 1. *For each $a \in \pi_1(M, p)$ we denote the corresponding deck transformation on $\tilde{M}$ by the capital letter A.* The fundamental vector field v_a on $\tilde{M}$, associated to a, is then defined as follows. For $q \in \tilde{M}$ let $\gamma: [0, 1] \to \tilde{M}$ be the unique geodesic with $\gamma(0) = q$ and $\gamma(1) = A(q)$. Then $v_a(q) = \vec{\gamma}(0)$. Note that if $a \neq 1$, then v_a is nowhere-zero.

Lemma 1. *At any point $q \in \tilde{M}$ the vectors*

$$\{v_a(q)/\|v_a(q)\| \in T_q(\tilde{M}) : 1 \neq a \in \pi_1(M, p)\}$$

are dense on the unit sphere in $T_q(\tilde{M})$. In particular, they are a spanning set for $T_q(\tilde{M})$.

Proof. Choose $X \in T_q(\tilde{M})$ with $\|X\| = 1$, and let $D \subset \tilde{M}$ be a compact fundamental domain for M which contains q. Then for any $t > 0$ there exists $a \in \pi_1(M, p)$ such that $A^{-1}(\exp_q(tX)) \in D$. Thus, if d is the diameter of D, we have that for each integer $n > 0$ there exists $a_n \in \pi_1(M, p)$ such that dist $(A_n(q), \exp_q(nX)) \leq d$. Letting v_n denote $v_{a_n}(q)$, we then see from Fact 4 that

$$\lim_{n\to\infty} \frac{\langle X, v_n\rangle}{\|X\|\,\|v_n\|} = 1 ,$$

and the lemma is proved.

Lemma 2. *For any $a, b \in \pi_1(M, p)$ we have $A^*v_b = v_{aba^{-1}}$. Thus, if $ab = ba$, the vector field v_b is A-invariant.*

Proof. Fix any $q \in \tilde{M}$. Then $q = A(q')$ for some $q' \in \tilde{M}$. Let $\gamma(s): 0 \leq s \leq 1$ be the unique geodesic from q' to $B(q')$. Then $A\gamma(s)$ is the unique geodesic from q to $ABA^{-1}(q)$. Hence $(A_*v_b)(q) = A_*\vec{\gamma}(0) = (\overrightarrow{A\gamma})(0) = v_{aba^{-1}}(q)$, and the lemma is proved.

Remark. Lemma 2 provides a simple proof of Gottlieb's theorem: *If $\chi(\pi)$ $(=\chi(K(\pi, 1))) \neq 0$, then π has no center*, for groups π which are fundamental groups of compact manifolds of nonpositive curvature. In § 7 this is generalized to fundamental groups of complete manifolds of nonpositive curvature and finite volume.

2. The flat torus theorem

In this section we prove the following

Theorem 1. *Let M be a compact C^∞ riemannian manifold of nonpositive curvature. Then there exists an abelian subgroup of rank k in $\pi_1(M, *)$ if and only if there exists a flat k-torus immersed isometrically and totally geodesically in M.*

Proof. Let $f: T \to M$ be a totally geodesic immersion where T is a torus. Then the induced map $f_*: \pi_1(T) \to \pi_1(M)$ is injective. To see this let $1 \neq a \in \pi_1(M)$, and choose a geodesic loop $\gamma \in a$. (See Fact 3.) Then $f \circ \gamma$ is a geodesic loop in M, and by Fact 1 no geodesic loop in M can be null-homotopic. This proves one half of the theorem.

To prove the converse we proceed as follows. For fixed $p_0 \in M$ let $\mathscr{A} \subset \pi_1(M, p_0)$ be an abelian subgroup of rank k with generators $a_1, \cdots, a_k$. (Recall that $\mathscr{A}$ is free abelian [8, p. 103].) For each $p \in M$ let $a_1', \cdots, a_k' \in \pi_1(M, p)$ be the images of $a_1, \cdots, a_k$ under some path isomorphism (corresponding to a path in M from p_0 to p), and define $f(p) = \inf \{\text{length}(\gamma): \gamma: [0,1] \to M$ is a piecewise smooth curve such that for some $g \in \pi_1(M, p), \gamma | [(i-1)/k, i/k] \in g a_i' g^{-1}$ for $i = 1, \cdots, k\}$. Then f is a well defined continuous function on M and therefore achieves a minimum at some point $q \in M$. Furthermore, from Fact 3 and the fact that the deck transformations are properly discontinuous we know that there exist a $g \in \pi_1(M, q)$ and geodesic loops $\gamma_1, \cdots, \gamma_k$ at q with $\gamma_i \in g a_i' g^{-1}$ for $i = 1, \cdots, k$ such that $f(q) = \text{length}(\gamma_1) + \cdots + \text{length}(\gamma_k)$. For convenience we relabel each element $g a_i' g^{-1} \in \pi_1(M, q)$ simply by a_i.

We now claim that each γ_i is smoothly closed. Let $\pi: \tilde{M} \to M$ be the universal riemannian covering of M and fix $\tilde{q} \in \tilde{M}$ as the distinguished point above q. Let $A_1, \cdots, A_k$ be the deck transformations of $\tilde{M}$ corresponding to $a_1, \cdots, a_k$ respectively. For each i let $\tilde{\gamma}_i$ be the lift of γ_i to a curve emanating from $\tilde{q}$. Then $\tilde{\gamma}_i$ is the unique geodesic in $\tilde{M}$ from $\tilde{q}$ to $A_i(\tilde{q})$. We shall now show that the geodesic γ_1 is smoothly closed. Suppose it is not. Then the unique broken geodesic from $\tilde{q}$ to $A_1(\tilde{q})$ to $A_1^2(\tilde{q})$ $(= \tilde{\gamma}_1 \cup A_1(\tilde{\gamma}_1))$ is not smooth. Call this geodesic $\tilde{\gamma}$. Let $x \in \tilde{\gamma}$ be a point lying between $\tilde{q}$ and $A_1(\tilde{q})$ but not equal to $\tilde{q}$ or $A_1(\tilde{q})$. Then

$$\begin{aligned} d(x, A_1(x)) &< d(x, A_1(\tilde{q})) + d(A_1(\tilde{q}), A_1(x)) \\ &= d(x, A_1(\tilde{q})) + d(\tilde{q}, x) = d(\tilde{q}, A_1(\tilde{q})) , \end{aligned}$$

where d denotes the distance function on $\tilde{M}$. Now, let $\tilde{\gamma}_1$ be parametrized over $[0, 1]$. Then, by [2, Lemma 10.1] the function $t \mapsto d(\tilde{\gamma}_1(t), A_i\tilde{\gamma}_1(t))$ is a convex function on $[0, 1]$ for each i. However, $d(\tilde{\gamma}_1(0), A_i\tilde{\gamma}_1(0)) = d(\tilde{q}, A_i(\tilde{q})) = d(A_1(\tilde{q}), A_1 A_i(\tilde{q})) = d(A_1(\tilde{q}), A_i A_1(\tilde{q})) = d(\tilde{\gamma}_1(1), A_i\tilde{\gamma}_1(1))$. Thus, for all $t \in [0, 1]$ we have $d(\tilde{\gamma}_1(t), A_i\tilde{\gamma}_1(t)) \leq d(\tilde{q}, A_i\tilde{q})$. Hence, for $0 < t < 1$

$$\sum_{i=1}^{k} d(\tilde{\gamma}_1(t), A_i\tilde{\gamma}_1(t)) < \sum_{i=1}^{k} d(\tilde{q}, A_i(\tilde{q})) .$$

This contradicts the minimality of f at q, and therefore γ_1 is smoothly closed. By proceeding in the same way, one can show that each of the geodesics $\gamma_1, \cdots, \gamma_k$ is smoothly closed.

For each i, let $\bar{\gamma}_i$ be the infinite (i.e., maximal) geodesic in $\tilde{M}$ containing $\tilde{\gamma}_i$. Then $\bar{\gamma}_i$ is A_i-invariant, and since the elements $a_1, \cdots, a_k$ are independent in $\mathscr{A}$, no two geodesics $\bar{\gamma}_i, \bar{\gamma}_j$ with $i \neq j$ coincide. We shall now construct a complete flat totally geodesic $\mathscr{A}$-invariant k-dimensional submanifold of $\tilde{M}$ which contains $\bar{\gamma}_1 \cup \cdots \cup \bar{\gamma}_k$. We first observe that since $A_1A_2 = A_2A_1$, all the geodesics $A_2^k\bar{\gamma}_1$ for $k = 0, \pm 1, \pm 2, \cdots$ are of the same type. Thus by Proposition 1 for each integer $k > 0$ there exists a flat embedded totally geodesic ribbon of surface, say $\sum^k$, between $A_2^{-k}\bar{\gamma}_1$ and $A_2^k\bar{\gamma}_1$. Since $\bar{\gamma}_2$ is A_2-invariant and $A_1A_2 = A_2A_1$, we have $\bar{\gamma}_1 \subset \sum^k \subset \sum^{k+i}$ for all $k, i \geq 0$. It follows that $\sum_{1,2} = \bigcup_k \sum^k$ is an embedded totally geodesic A_1- and A_2-invariant surface in M, which is isometric to a flat 2-plane. (Observe that each infinite geodesic in $\sum_{1,2}$ which is parallel to $\bar{\gamma}_1$ (respt. $\bar{\gamma}_2$) is A_1-(respt. A_2-) invariant.)

For each $p \in \sum_{1,2}$ we denote by $\bar{\gamma}_p$ the unique geodesic passing through x and $A_3(x)$. We then define

$$\sum_{1,2,3} = \bigcup_{p \in \Sigma_{1,2}} \bar{\gamma}_p .$$

$\sum_{1,2,3}$ is invariant under A_1, A_2 and A_3 and contains the surfaces $\sum_{1,2}$, $\sum_{1,3}$ and $\sum_{2,3}$. From the commutativity and independence of a_1, a_2, a_3 and the methods developed above, it is straightforward to verify that $\sum_{1,2,3}$ is an embedded totally geodesic manifold which is isometric to a flat 3-plane and on which A_1, A_2 and A_3 act by translations.

Proceeding in this manner we eventually construct an embedded totally geodesic $\mathscr{A}$-invariant manifold $\sum_{1,\cdots,k}$ which is isometric to a flat k-plane and on which $\mathscr{A}$ acts by translations. This completes the proof.

A straightforward argument using the above techniques establishes the following corollary. (See [14].)

Corollary 2. *Let M be as in Theorem* 1. *Then every abelian subgroup of $\pi_1(M, *)$ is a finitely generated (free) group of rank* $\leq \dim(M)$, *where equality holds if and only if M is flat.*

3. The splitting theorem

We shall now establish

Theorem 2. *Let M be a compact real analytic riemannian manifold of nonpositive curvature, and suppose that the fundamental group of M splits as a direct product:*

$$\pi_1(M) = \mathscr{A}_1 \times \cdots \times \mathscr{A}_N .$$

If $\pi_1(M)$ has no center, then there exists a riemannian splitting of M,

$$M = M_1 \times \cdots \times M_N ,$$

such that $\pi_1(M_k) = \mathscr{A}_k$ for each k.

Proof. Suppose $\pi_1(M)$ has no center and $\pi_1(M) = \mathscr{A} \times \mathscr{B}$ where both $\mathscr{A}$ and $\mathscr{B}$ are nontrivial. We shall show that M is isometric to a riemannian product $M_{\mathscr{A}} \times M_{\mathscr{B}}$ where $\pi_1(M_{\mathscr{A}}) = \mathscr{A}$ and $\pi_1(M_{\mathscr{B}}) = \mathscr{B}$.

Fix a point $p \in M$ and let $\pi\colon \tilde{M} \to M$ be the corresponding universal covering (cf. Fact 1). Then for each $a \in \mathscr{A} \subset \pi_1(M)$ we have a corresponding deck transformation A of $\tilde{M}$, and we shall define the set $\tilde{M}_a$ to be the union of all infinite A-invariant geodesics in $\tilde{M}$.

Note that for each $a \in \mathscr{A}$, the set $\tilde{M}_a$ is nonempty. Indeed, if $a \neq 1$, there exists a smoothly closed geodesic $\gamma \subset M$ belonging to the free homotopy class of a. The various lifts of γ to $\tilde{M}$ are infinite geodesics translated by the various elements GAG^{-1}, for G in the deck group. In particular, one of the lifts is translated by A, and $\tilde{M}_a$ must be nonempty.

Since any two A-invariant geodesics are of the same type, the arguments of Proposition 1 and Corollary 1 show that $\tilde{M}_a$ is a complete connected totally geodesic submanifold of $\tilde{M}$.

Moreover, these arguments show that the vector field v_a is parallel along $\tilde{M}_a$, and, in fact, $\tilde{M}_a$ splits as $\boldsymbol{R} \times \tilde{M}'$ where for each $(t, x) \in \boldsymbol{R} \times \tilde{M}'$, $A(t, x) = (t + c_a, x)$ for some fixed constant c_a.

It follows that for some $a \in \mathscr{A}$, we have $\dim(\tilde{M}_a) < \dim(\tilde{M})$. Otherwise, $\tilde{M}_a = \tilde{M}$ for all $a \in \mathscr{A}$ and each of the vector fields v_a is globally parallel on $\tilde{M}$. The distribution spanned by these fields gives a riemannian splitting $\tilde{M} = \boldsymbol{R}^k \times \tilde{M}'$, where $\boldsymbol{R}^k$ is a flat euclidean k-space, and for each $a \in \mathscr{A}$ the corresponding deck transformation A is written as $A(\tau, x) = (\tau + c_a, x)$ for each $(\tau, x) \in \boldsymbol{R}^k \times \tilde{M}'$ and some fixed $c_a \in \boldsymbol{R}^k$. In particular, $\mathscr{A}$ is abelian, contradicting the assumption on the center of $\mathscr{A}$. For future reference we note this observation as

Fact A. *If $\tilde{M}_a = \tilde{M}$ for all $a \in \mathscr{A}$, then $\mathscr{A}$ is abelian, and there exists a splitting $\tilde{M} = \mathrm{R}^k \times \tilde{M}'$ where $\mathscr{A}$ fixes the second factor and operates on the first factor by translation.*

We now choose $a \in \mathscr{A}$ so that $\dim(\tilde{M}_a) < \dim(\tilde{M})$. Of course, for any $b \in \mathscr{B}$, we have $BA = AB$ and therefore $B(\tilde{M}_a) = \tilde{M}_a$. Hence $\tilde{M}_a$ is $\mathscr{B}$-invariant.

We claim that the set $\pi(\tilde{M}_a)$ in M is compact. To see this let $\{p_n\}$ be a sequence from $\pi(\tilde{M}_a)$ such that $p_n \to p \in M$. Through each p_n there passes a smooth closed geodesic (not necessarily embedded) of length $l = \operatorname{dist}(q, Aq)$ where q is any point of $\tilde{M}_a$. Denote this geodesic by $\gamma_n(s)$; $0 \leq s \leq l$. Let

$e_n = \vec{\gamma}_n(0)$. By passing to a subsequence we may assume that the sequence $\{e_n\}$ converges in the tangent sphere bundle of M to a vector $e \in T_p(M)$. Then the geodesic $\gamma_\infty(s) = \exp_p(se)$ is a smooth closed geodesic. Since the convergence of γ_n to γ_∞ is uniform, there is some integer N such that each γ_n, for $n \geq N$, represents essentially the same bound homotopy class (bound, say, at p by moving $\gamma_n(0)$ slightly). Let U be a neighborhood of p which has an even covering in $\tilde{M}$, and by possibly raising N assume $\gamma_n(0) \in U$ for all $n \geq N$. Let $\tilde{U}$ be any component of the even cover of U, and lift $\{p_n\}_{n \geq N}$ and p to the sequence $\tilde{p}_n \to \tilde{p}$ in $\tilde{U}$. Then each γ_n, $N \leq n \leq \infty$, lifts to an infinite geodesic $\tilde{\gamma}_n$ through $\tilde{p}_n$, and all these geodesics are invariant under *the same* deck transformation ($= GAG^{-1}$ for some G) since they are smoothly closed and lie in the same bound homotopy class. In particular, if we choose $\tilde{U}$ so that $\tilde{\gamma}_N$ is A-invariant, then each $\tilde{\gamma}_n$, $N \leq n \leq \infty$, is A-invariant. Thus $\tilde{p}_n, \tilde{p} \in \tilde{M}_a$, and so $p \in \pi(\tilde{M}_a)$ and $\pi(\tilde{M}_a)$ is compact.

To recapitulate, we have proven

Fact B. *Let $\{p_n\}$ be any sequence in $\pi(\tilde{M}_a)$. Then there exists a subsequence $\{p_{n_i}\}$ such that $p_{n_i} \to p \in \pi(\tilde{M}_a)$ and such that if the lift of any p_{n_i} (or p) to $\tilde{p}_{n_i}$* (respt. $\tilde{p}$) *in $\tilde{M}$ lies in $\tilde{M}_a$, then the lift of the whole sequence to the same neighborhood of $\tilde{M}$ also lies in $\tilde{M}_a$.*

Suppose now that for some $a_1 \in \mathscr{A}$ we have $A_1(\tilde{M}_a) \cap \tilde{M}_a \neq \emptyset$. Then $A_1(\tilde{M}_a) = \tilde{M}_{a_i a a_i^{-1}}$ (the manifold of $A_1 A A_1^{-1}$-invariant geodesics) and we can apply the above argument to show that $\pi[A_1(\tilde{M}_a)]$ is compact. It is clear that $A_1(\tilde{M}_a) \cap \tilde{M}_a$ is a totally geodesic $\mathscr{B}$-invariant submanifold of $\tilde{M}$. Furthermore, $\pi[A_1(\tilde{M}_a) \cap \tilde{M}_a]$ is compact. To see this let $\{p_n\}$ be a sequence in this set, and choose successive subsequences by using Fact B first for $\tilde{M}_a$ and then for $A_1(\tilde{M}_a)$. The final sequence lifts to a convergent subsequence in $A_1(\tilde{M}_a) \cap \tilde{M}_a$, and compactness follows.

It is clearly possible to choose elements $a_1, \cdots, a_q \in \mathscr{A}$ such that the totally geodesic $\mathscr{B}$-invariant manifold $\tilde{N} = \bigcap_{i=1}^q A_i(\tilde{M}_a) \cap \tilde{M}_a$ has the following property. For any $c \in \pi_1(M)$ either $C(\tilde{N}) = \tilde{N}$ or $C(\tilde{N}) \cap \tilde{N} = \emptyset$. Thus $N \stackrel{\text{def}}{=\!=} \pi(\tilde{N})$ is an embedded totally geodesic submanifold which by the above argument is compact in the relative topology. We assert, moreover, that N is *topologically embedded*, that is, for every point $\tilde{p} \in \tilde{N}$ there is a neighborhood $\tilde{U}$ of $\tilde{p}$ in $\tilde{M}$, for which $\pi | \tilde{U}$ is a diffeomorphism into M, such that

$$\pi(\tilde{N} \cap \tilde{U}) = N \cap \pi(\tilde{U}) .$$

If this were not so, there would exist a $\tilde{p}$ and $\tilde{U}$ and a sequence of points $\{\tilde{p}_n\}$ in $\tilde{U} \cap \pi^{-1}(N)$ such that $\tilde{p}_n \to \tilde{p}$ and no two distinct $\tilde{p}_n$ lie in the same component of $\pi^{-1}(N)$. (Note that $\pi^{-1}(N) = \bigcup \{C(\tilde{N}) : c \in \pi_1(M)\}$ and for all C_1, C_2 either $C_1(\tilde{N}) = C_2(\tilde{N})$ or $C_1(\tilde{N}) \cap C_2(\tilde{N}) = \emptyset$.) However, by applying Fact B to the sequence $\{\pi(\tilde{p}_n)\}$ we obtain a contradiction. Thus N is a compact totally geodesic submanifold of M. Finally, we have that $N = \tilde{N}/G$ where

$G = \{c \in \pi_1(M) : C(\tilde{N}) = \tilde{N}\}$ and, clearly, $G = \mathscr{A}' \times \mathscr{B}$ for some subgroup $\mathscr{A}' \subset \mathscr{A}$.

We now focus attention on the manifold N. Observe that by Fact A we may assume that the group $\mathscr{A}'$ is abelian and the universal covering space $\tilde{N}$ of N splits as $\boldsymbol{R}^k \times \tilde{N}'$ where for each $a \in \mathscr{A}'$ the corresponding deck transformation is a simple translation of the first factor. If this were not the case, there would exist $a \in \mathscr{A}'$ such that $\dim(\tilde{N}_a) < \dim(\tilde{N})$, and we would repeat the process to obtain a new manifold $N_1 \subset N \subset M$ of smaller dimension. Consequently after a finite number of steps we would arrive at the desired situation.

For later use we shall analyse the action of $\mathscr{A}' \times \mathscr{B}$ on $\tilde{N}$. Let $a \in \mathscr{A}'$. Then there exists a vector $c_a \in \boldsymbol{R}^k$ such that $A(t, x) = (t + c_a, x)$ for each $(t, x) \in \boldsymbol{R}^k \times \tilde{N}' = \tilde{N}$. The vectors c_a span $\boldsymbol{R}^k$. Hence for each $b \in \mathscr{B}$ there are a vector $c_b \in \boldsymbol{R}^k$ and an isometry $B_1 : \tilde{N}' \to \tilde{N}'$ such that $B(t, x) = (t + c_b, B_1(x))$. Note that since $N = \tilde{N} / \mathscr{A}' \times \mathscr{B}$ is compact, so is the set $\tilde{N}' / \mathscr{B}$ where $\mathscr{B}$ acts by the projected action $b \to B_1$. Furthermore, from Lemma 1 we see that at any point $x \in \tilde{N}'$, the normalized fundamental vector fields for this projected action are dense in the unit sphere of $T_x(\tilde{N}')$. As a consequence we have the following

Fact C. *Let $S \subset \tilde{N}$ be any $\mathscr{B}$-invariant geodesically convex set.*

i) *Then for any translation $T(t, x) = (t + c, x)$, the set $T(S)$ is again $\mathscr{B}$-invariant and convex.*

ii) *Let $pr : \tilde{N} \to \tilde{N}'$ be the natural projection. Then $pr(S) = \tilde{N}'$.*

We shall now construct a totally geodesic $\mathscr{B}$-invariant submanifold $\tilde{M}_{\mathscr{B}} \subset \tilde{N}$ such that $\dim(\tilde{M}_{\mathscr{B}}) = \dim(\tilde{N}) - k$. Fix a point $p \in \tilde{N} \subset \tilde{M}$ and let $\langle \mathscr{B} p \rangle$ denote the smallest closed $\mathscr{B}$-invariant geodesically convex set containing p. Of course, $\langle \mathscr{B} p \rangle \subset \tilde{N}$. Now for any point $x \in \tilde{N}'$ consider the convex set $C_x = \langle \mathscr{B} p \rangle \cap \boldsymbol{R}^k \times \{x\}$. We assert that C_x is a bounded set. To see this we proceed as follows. Suppose C_x is unbounded. Then there exists a sequence of points $\{q_j\}_{j=1}^{\infty}$ in $\langle \mathscr{B} p \rangle \cap \boldsymbol{R}^k \times \{x\}$ such that $\|q_j\| \to \infty$. Let F be a compact fundamental domain for the action of $\mathscr{A}'$ on $\boldsymbol{R}^k \times \{x\}$. Then there exists a sequence of points $\{p_j\}$ from F and a sequence of translations $\{a_j\}_{j=1}^{\infty}$ from $\mathscr{A}'$ such that $A_j p_j = q_j$ for each j. Furthermore, since $\|q_j\| \to \infty$ and the $\|p_j\|$'s are bounded, we may assume, after passing to a subsequence, that the elements a_j are mutually distinct.

We now choose any element $a \in \mathscr{A}$, and consider the function $d(p', Ap') = \operatorname{dist}(p', Ap')$ defined for points $p' \in \tilde{M}$. This function is constant on the set $\mathscr{B} p = \{B(p) : b \in \mathscr{B}\}$, and is bounded on the set $\langle \mathscr{B} p \rangle$ since it is a convex function [2]. It follows that the sequence $d(q_j, Aq_j) = d(A_j p_j, AA_j p_j) = d(p_j, A_j^{-1} AA_j p_j)$ is bounded for all j. Consequently, by passing to subsequences we see that there exist points $p_0, q_0 \in \tilde{M}$ such that $p_j \to p_0$ and $A_j^{-1} AA_j(p_j) \to q_0$. Moreover, since

$$d(q_0, A_j^{-1}AA_jp_0) \le d(q_0, A_j^{-1}AA_jp_j) + d(A_j^{-1}AA_jp_j, A_j^{-1}AA_jp_0)$$
$$= d(q_0, A_j^{-1}AA_jp_j) + d(p_j, p_0) \to 0 \ ,$$

we have that $A_j^{-1}AA_j(p_0) \to q_0$. Therefore, since the deck group acts properly discontinuously, there is a constant J_a such that for $j, i \ge J_a$

$$A_j^{-1}AA_j = A_j^{-1}AA_i \ .$$

From the fact that $\mathscr{A}$ is finitely generated it then follows that there exists a constant J such that for $j > i \ge J$ the nontrivial element $A_jA_i^{-1}$ lies in the center of $\pi_1(M)$. This contradicts our assumption that $\pi_1(M)$ has no center, and the assertion is established.

We now claim that for each x, C_x consists of exactly one point. Suppose not. Then since $C_x \subset \boldsymbol{R}^k \times \{x\}$ is a compact convex set in euclidean space, given any $\bar{p} \in C_x$ there would exist a translation T of $\boldsymbol{R}^k \times \{x\}$ such that $\bar{p} \in C_x \cap T(C_x) \subsetneq C_x$. Extending T naturally to $\tilde{N} = \boldsymbol{R}^k \times \tilde{N}'$ we would have by Fact C that $\langle \mathscr{B}p \rangle \cap T(\langle \mathscr{B}p \rangle)$ is a proper closed $\mathscr{B}$-invariant convex subset of $\langle \mathscr{B}p \rangle$ which contains $\bar{p}$. However, in the special case $x = x_0$ where $p = (t_0, x_0)$, if we set $\bar{p} = p$, we would obtain a contradiction to the minimality of $\langle \mathscr{B}p \rangle$. Thus $C_{x_0} = \{p\}$. In the case of any other x, we would have

$$\langle \mathscr{B}p \rangle \cap T(\langle \mathscr{B}p \rangle) \cap \boldsymbol{R}^k \times \{x_0\} = \{p\} \cap T(\{p\}) = \emptyset \ ,$$

which contradicts (ii) of Fact C.

It follows that $\langle \mathscr{B}p \rangle$ is the graph of a function $f\colon \tilde{N}' \to \boldsymbol{R}^k$ and therefore $\tilde{M}_{\mathscr{B}} \overset{\text{def}}{=\!=} \langle \mathscr{B}p \rangle$ is a totally geodesic $\mathscr{B}$-invariant manifold without boundary and of dimension $= \dim(\tilde{N}')$.

Furthermore, the manifold $\tilde{M}_{\mathscr{B}}/\mathscr{B}$ is compact. In fact, let $F_0 \subset \tilde{N}'$ be a compact fundamental domain for the projected action of $\mathscr{B}$ or $\tilde{N}'$, and let $pr\colon \tilde{M}_{\mathscr{B}} \to \tilde{N}'$ be the projection. Then $F = pr^{-1}(F_0)$ is a compact fundamental domain for $\mathscr{B}$ on $\tilde{M}_{\mathscr{B}}$.

We can now construct a foliation of $\tilde{M}$ by totally geodesic $\mathscr{B}$-invariant manifolds parallel to $\tilde{M}_{\mathscr{B}}$. To begin, let $\gamma \subset \tilde{M}_{\mathscr{B}}$ be any infinite geodesic and fix any $a \in \mathscr{A}$. Then the geodesics γ and $A\gamma$ are of the same type. To see this let $F \subset \tilde{M}_{\mathscr{B}}$ be a compact fundamental domain, and let $\delta = \max\{d(p, A_p) : p \in F\}$. Then for each value of t there exists $b \in \mathscr{B}$ such that $B(\gamma(t)) \in F$. Consequently, $d(\gamma(t), A\gamma(t)) = d(B\gamma(t), BA\gamma(t)) = d(B\gamma(t), AB\gamma(t)) \le \delta$, and the geodesics are of the same type. Thus by Corollary 1 and analyticity there is a flat totally geodesic surface through γ and $A\gamma$. There are two important consequences of this:

(1) Each geodesic $\gamma_t(s) = \exp_{\gamma(s)}(tv_a)$ for $t \in R$ is of the same type as γ.

(2) By considering all geodesics $\gamma \subset \tilde{M}_{\mathscr{B}}$ we have that v_a is globally parallel along $\tilde{M}_{\mathscr{B}}$, i.e., for all tangent vector fields X on $\tilde{M}_{\mathscr{B}}$

$$(3.1) \qquad \nabla_X v_a = 0 .$$

Furthermore, the sectional curvature

$$(3.2) \qquad \langle R_{v_a,X} v_a, X\rangle \equiv 0 .$$

From (1) it is easy to see that for any $t \in \boldsymbol{R}$ and any $a \in \mathscr{A}$ the manifold $\tilde{M}_{\mathscr{B},t,a} = \{\exp_x(tv_a) : x \in \tilde{M}_{\mathscr{B}}\}$ is totally geodesic and $\mathscr{B}$-invariant. Moreover, the map $\exp: \tilde{M}_{\mathscr{B}} \to \tilde{M}_{\mathscr{B},t,a}$ is a $\mathscr{B}$-equivariant isometry. Any two such manifolds $\tilde{M}_{\mathscr{B},t,a}$ either are disjoint or coincide since the tangent space at any point $p \in \tilde{M}_{\mathscr{B},t,a}$ is simply the span of $\{v_b(p)\}_{b\in\mathscr{B}}$.

By Lemma 1 the union of the manifolds $\tilde{M}_{\mathscr{B},t,a}$ is dense in $\tilde{M}$. Moreover, since each $\tilde{M}_{\mathscr{B},t,a}/\mathscr{B}$ is compact we may repeat our game of exponentiation there. It follows easily that through every point $p \in \tilde{M}$ there passes a unique totally geodesic $\mathscr{B}$-invariant submanifold $\tilde{M}_{\mathscr{B},p}$ which is $\mathscr{B}$-equivariantly isometric and geodesically parallel to $\tilde{M}_{\mathscr{B}}$. We thus have our desired foliation. (Note that at any $p \in \tilde{M}$, the tangent plane to the distribution is span $\{v_b(p)\}_{b\in\mathscr{B}}$.)

We may now repeat the process to obtain a similar foliation of $\tilde{M}$ by $\mathscr{A}$-invariant manifolds (whose tangent planes are spanned by $\{v_a\}_{a\in\mathscr{A}}$).

From (2) above we see that for any $a \in \mathscr{A}$ and $b \in \mathscr{B}$

$$(3.3) \qquad \nabla_{v_a} v_b \equiv \nabla_{v_b} v_a \equiv 0 ,$$

$$(3.4) \qquad \langle R_{v_a,v_b} v_a, v_b\rangle \equiv 0 .$$

Furthermore we claim that

$$(3.5) \qquad \langle v_a, v_b\rangle \equiv 0 .$$

Suppose this were not the case. Then the orthogonal projection of some v_a onto $\tilde{M}_{\mathscr{B}}$ would be a globally paralled $\mathscr{B}$-invariant nonzero tangent vector field on $\tilde{M}_{\mathscr{B}}$. However, by Theorem 3 (§ 5) this implies $\mathscr{B}$ has a nontrivial center.

We can now express $\tilde{M} = \tilde{M}_{\mathscr{A}} \times \tilde{M}_{\mathscr{B}}$ (a riemannian product) where $\mathscr{A}$ acts only on the first factor and $\mathscr{B}$ only on the second, and so $M = (\tilde{M}_{\mathscr{A}}/\mathscr{A}) \times (\tilde{M}_{\mathscr{B}}/\mathscr{B})$ as claimed.

4. On the splitting theorem for fundamental groups with center

In the case that $\pi = \mathscr{A} \times \mathscr{B}$ and π has a nontrivial center, the splitting theorem as stated is not true. To see this, let Σ be a compact surface of genus $g > 1$ and constant negative curvature, and consider the homology covering $\tilde{\Sigma} \to \Sigma$ with deck group $D \cong 2g\boldsymbol{Z}$ (and $\pi_1\tilde{\Sigma} = [\pi_1\Sigma, \pi_1\Sigma]$). Choosing a basis $a_1, \cdots, a_{2g}$ for D we can construct an action of D (by isometries) on a flat rectangular torus $T^{2g} = S^1 \times \cdots \times S^1$ by letting a_k rotate the k-th factor through some irrational angle. The diagonal action of D on $T^{2g} \times \tilde{\Sigma}$ is certainly free and properly discontinuous, since the action on $\tilde{\Sigma}$ is. The manifold

$$M \stackrel{\text{def}}{=\!=} (T^{2g} \times \tilde{\Sigma})/D$$

is compact nonpositively curved and has fundamental group $\pi_1(M) \cong 2g\mathscr{Z} \times \pi_1(\Sigma)$. However, there is no global riemannian splitting of M (nor is this true of any finite covering space). One way to see this is to observe that on the universal covering space $\tilde{M}$ of M the fundamental vector fields which correspond to $\pi_1(\Sigma)$ actually span the tangent space at any point.

Note, of course, that M is diffeomorphic to a product. It can be expressed as a flat torus bundle over Σ where $\pi_1(\Sigma)$ acts on the torus fiber by rotations. Thus the bundle is differentiably, but not metrically, trivial.

This example is indicative of what happens in the general case. Let M be a compact nonpositively curved manifold with fundamental group $\pi = \mathscr{A} \times \mathscr{B}$, and let $\mathscr{Z}_\pi = \mathscr{Z}_\mathscr{A} = \mathscr{Z}_\mathscr{A} \times \mathscr{Z}_\mathscr{B}$ be the center of π. Then $\mathscr{Z}_\mathscr{A} = \boldsymbol{Z}^a$ and $\mathscr{Z}_\mathscr{B} = \boldsymbol{Z}^b$ for some integers a and b, and the universal covering $\tilde{M}$ of M splits as a riemannian product $\boldsymbol{R}^a \times \boldsymbol{R}^b \times \tilde{M}_0$, where $\mathscr{Z}_\mathscr{A}$ acts by translations of $\boldsymbol{R}^a$, $\mathscr{Z}_\mathscr{B}$ acts by translations of $\boldsymbol{R}^b$, and, in general, any $g \in \pi$ acts by a translation of $\boldsymbol{R}^a \times \boldsymbol{R}^b$ and an isometry of M_0, [12].

Consider the projected action of $\pi/\mathscr{Z}_\pi = \mathscr{A}' \times \mathscr{B}'$ where $\mathscr{A}' = \mathscr{A}/\mathscr{Z}_\mathscr{A}$ and $\mathscr{B}' = \mathscr{B}/\mathscr{Z}_\mathscr{B}$ on the manifold $\tilde{M}_0$. This action is properly discontinuous and has compact quotient. Using the above techniques it is not hard to show that there exists a riemannian splitting $\tilde{M}_0 = \tilde{M}_\mathscr{A} \times \tilde{M}_\mathscr{B}$ such that $\mathscr{A}'$ acts only on the first factor and $\mathscr{B}'$ only on the second. (First construct $\tilde{N} = \boldsymbol{R}^a \times \boldsymbol{R}^b \times \tilde{N}_0 \subset \tilde{M}$ and then use the fact that the effective action on $\tilde{M}_0 \supset \tilde{N}_0$ is centerless and properly discontinuous.)

It follows that we have a covering of M by a riemannian product

$$(T^a \times M_\mathscr{A}) \times (T^b \times M_\mathscr{B}) \to M$$

(T^s being a flat s-torus) with covering group $\mathscr{A}' \times \mathscr{B}'$ where $\mathscr{A}'$ acts on the second factor only by rotations of the torus T^b, and $\mathscr{B}'$ acts on the first factor only by rotations of T^a.

Therefore, as in the example, the manifold M is diffeomorphic to a product, and locally there exists a riemannian splitting. However, globally there may be some twisting along the flat foliations arising from the center of π. In analogy with flat manifold theory, we say that M is a toral extension of $(T^a \times M_\mathscr{A})/\mathscr{A}'$ by $(T^b \times M_\mathscr{B})/\mathscr{B}'$.

5. On parallel vector fields and isometries

Recall that a vector field v on a riemannian manifold M is said to be *globally parallel* (or simply *parallel*) if for all $p \in M$ and all $X \in T_p(M)$ we have $\nabla_X v = 0$.

The integral curves of such a field are all geodesics and, if the curvature of M is everywhere nonpositive, then every such vector field is harmonic.

Proposition 2. *Let M be a compact riemannian manifold of nonpositive curvature, and suppose that there exists a nontrivial isometry $f: M \to M$ which is homotopic to the identity map. Then there exists a globally parallel vector field v on M, and the flow generated by v is a 1-parameter group of isometries which contains f.*

Proof. Let $\pi: \tilde{M} \to M$ be the universal riemannian covering of M, and $\tilde{f}: \tilde{M} \to \tilde{M}$ be a lifting of f to $\tilde{M}$. Since f is homotopic to the identity, we have that $\tilde{f}$ commutes with every deck transformation of the covering. We now define a vector field $\tilde{v}$ on $\tilde{M}$ as follows. For each $p \in \tilde{M}$ let $\gamma_p(t)$, $0 \leq t \leq 1$, be the (unique) geodesic in $\tilde{M}$ from p to $\tilde{f}(p)$, and set $\tilde{v}_p = (d\gamma_p/dt)(0)$. We assert that v is globally parallel on $\tilde{M}$. To see this we first note that by [2, § 4.2] the function $\phi(p) = d^2(p, \tilde{f}(p))$, where d is the distance function on $\tilde{M}$, is a convex function on $\tilde{M}$ which is left invariant by every deck transformation. Thus $\tilde{\phi}$ projects to a convex function ϕ on M which by the compactness of M must be constant [2, § 2.2]. We now consider any $p \in \tilde{M}$ and $X \in T_p(\tilde{M})$, and let $\gamma(t)$, $-\infty < t < \infty$, be the unique geodesic in M with $(d\gamma/dt)(0) = X$. Then since $d(\gamma(s), \tilde{f} \circ \gamma(s)) =$ constant, the geodesics γ and $\tilde{f} \circ \gamma$ are of the same type. Thus it follows from Proposition 1 that either $\tilde{f}$ translates the geodesic γ or there exists a flat totally geodesic ribbon of surface between γ and $\tilde{f} \circ \gamma$ to which $\tilde{v} | \gamma$ is everywhere tangent. In either case we have $(\nabla_X \tilde{v})_p = 0$ and the assertion is proved.

Let D be any deck transformation of the covering. Then since $D\tilde{f} = \tilde{f}D$, we have that $D_* \tilde{v} = \tilde{v}$. Thus the vector field $\tilde{v}$ projects to a well defined globally parallel vector field v on M. Since $\tilde{f}$ is nontrivial, v is nonzero. The rest of the proposition now follows easily.

We now establish the following.

Theorem 3. *Let M be a compact riemannian manifold of nonpositive curvature. Then there exist exactly k linearly independent globally parallel vector fields on M if and only if the center of $\pi_1(M)$ has rank k.*

Proof. It follows from the center theorem that if the center of $\pi_1(M)$ has rank k, then there exist k globally parallel linearly independent vector fields on M. For the converse we shall suppose $v_1, \cdots, v_k$ form a maximal set of linearly independent parallel vector fields on M and show that the rank of the center is $\geq k$. Let $\pi: \tilde{M} \to M$ be the universal riemannian covering of M, and $\tilde{v}_1, \cdots, \tilde{v}_k$ be the lifts of $v_1, \cdots, v_k$ to $\tilde{M}$. Let $\Pi \simeq \pi_1(M)$ be the group of deck transformations of $\tilde{M}$. Then for each $G \in \Pi$, $G_* \tilde{v}_j = \tilde{v}_j$ for $i = 1, \cdots, k$. It is not difficult to see that the distribution span$\{\tilde{v}_1, \cdots, \tilde{v}_k\}$ is involutive and its integral leaves determine a riemannian product $\tilde{M} = \boldsymbol{R}^k \times \tilde{M}'$ where $\boldsymbol{R}^k$ is a flat euclidean k-space. Furthermore, we must have that for each $G \in \Pi$,

$$G(x, y) = (x + c_G, g(y)) \tag{5.1}$$

for all $(x, y) \in \boldsymbol{R} \times \tilde{M}'$, where $c_G \in \boldsymbol{R}^k$ and $g: \tilde{M}' \to \tilde{M}'$ is an isometry. (From now on, for any deck transformation G the component of G on the noneuclidean factor will always be denoted by the corresponding small letter g.) Let $\mathscr{Z}$ denote the center of Π and note that $Z \in \mathscr{Z}$ if and only if $Z(x, y) = (x + c_z, y)$ for all $(x, y) \in \boldsymbol{R}^k \times \tilde{M}'$. Our first assertion is the following.

Lemma 3. *Let $G \in \Pi$, and suppose that there exists some $(x_0, y_0) \in \boldsymbol{R}^k \times \tilde{M}'$ such that $G(x_0, y_0) = (x_0 + c_G, y_0)$. Then there exists an integer $n \geq 1$ such that $G^n \in \mathscr{Z}$.*

Proof. Let H be any element of Π, and consider the transformations $HG^nH^{-1}G^{-n} \in \Pi$. For all n we have

$$\begin{aligned} d((x_0, y_0), HG^nH^{-1}G^{-n}(x_0, y_0)) &= d((x_0, y_0), (x_0, hg^nh^{-1}(y_0))) \\ &= d((x_0, h^{-1}(y_0)), (x_0, g^nh^{-1}(y_0))) \\ &\leq d((x_0, h^{-1}(y_0)), (x_0, y_0)) + d((x_0, y_0), (x_0, g^nh^{-1}(y_0)) \\ &= 2d((x_0, y_0), (x_0, h^{-1}(y_0))) \ , \end{aligned}$$

where d denotes the distance function on $\tilde{M}$. Since Π acts properly discontinuously, there must be integers p, q such that $p > q$ and $HG^pH^{-1}G^{-p}(x_0, y_0) = HG^qH^{-1}G^{-q}(x_0, y_0)$. Since Π acts freely, we then have that $HG^pH^{-1}G^{-p} = HG^qH^{-1}G^{-q}$, i.e., $HG^mH^{-1}G^{-m} = 1$ where $m = p - q$. The lemma now follows from the fact that Π is finitely generated.

We now observe that if v is a parallel vector field on M, then v must be a Killing vector field. Choose any $p \in M$ and any $X, Y \in T_p(M)$. Extend X and Y to local fields $\tilde{X}$ and $\tilde{Y}$ by parallel translation along geodesic rays emanating from p. Then we have $[v, \tilde{X}]_p = (\nabla_v \tilde{X})_p = 0$, and similarly $[v, \tilde{Y}]_p = 0$. Hence for the riemannian metric g on M we have that

$$\begin{aligned} 0 &= v_p g(\tilde{X}, \tilde{Y}) \\ &= (\mathscr{L}_v g)_p(X, Y) + g_p([v, \tilde{X}]_p, Y) + g_p(X, [v, \tilde{Y}]_p) \\ &= (\mathscr{L}_v g)_p(X, Y) \ , \end{aligned}$$

where $\mathscr{L}_v$ denotes Lie differentiation. Thus $\mathscr{L}_v g \equiv 0$, and v is a Killing vector field.

Let $I(M)$ denote the group of isometries of M, and let $I(M)^0$ denote the identity component. By the above, every parallel vector field gives rise to a 1-parameter group in $I(M)$, and by Proposition 2 every nontrivial element of $I(M)^0$ gives rise to a nonzero parallel vector field on M. Thus the Lie algebra of $I(M)$ is naturally isomorphic to the space of parallel vector fields on M. Since for any two parallel vector fields v, w we have $[v, w] = \nabla_v w - \nabla_w v = 0$, the group $I(M)^0$ must be a k-torus. Furthermore, it is clear that $I(M)^0$ acts locally freely on M and each orbit of the action is a flat totally geodesic k-torus.

Let T_0 be any such orbit, and let $\boldsymbol{R}_0^k = \boldsymbol{R}^k \times \{p_0\} \subset \boldsymbol{R}^k \times \tilde{M}'$ be the inverse image of T_0 in the universal covering space. We now consider the subgroup

$$\Pi_0 = \{G \in \Pi : G(\boldsymbol{R}_0^k) = \boldsymbol{R}_0^k\} .$$

Since Π_0 acts freely and properly discontinuously by isometries on $\boldsymbol{R}_0^k$ and since $\boldsymbol{R}_0^k/\Pi_0 = T_0$, we have that $\Pi_0 \simeq \boldsymbol{Z} \times \cdots \times \boldsymbol{Z}$ (k-times). Let $A_1, \cdots, A_k$ be a set of generators for Π_0. Then by Lemma 3 there exist integers $n_1, \cdots, n_k$ such that $A_i^{n_i} \in \mathscr{Z}$ for $i = 1, \cdots, k$. Let $m = n_1 \times \cdots \times n_k$, and set $\Pi_0^m = \{A^m : A \in \Pi_0\}$. Then Π_0^m is a free abelian group of rank k which is contained in $\mathscr{Z}$. Thus rank $(\mathscr{Z}) \geq k$ and the theorem follows immediately.

6. The Bochner-Frankel-Hurwitz theorem

It is known that if M is a compact manifold with strictly negative curvature, then the only isometry of M which is homotopic to the identity ($\cong \mathbf{1}$) is the identity itself [5]. As an immediate consequence of our previous discussion, we now have a generalization of this theorem for manifolds of nonpositive curvature.

Theorem 4. *Let M be a compact riemannian manifold of nonpositive curvature, and suppose that there exists a nontrivial isometry f of M such that $f \cong \mathbf{1}$. Then:*

(a) *M admits a globally parallel nonzero vector field,*

(b) *the center of $\pi_1(M)$ is nontrivial (and thus the conclusions of the center theorem hold),*

(c) *there exists a torus T^k acting locally freely* (i.e., *the isotropy subgroup of every point is finite) on M by isometries, and $f \in T^k$.*

Corollary 3. *Let M be as in Theorem* 4, *and suppose that any of the following conditions holds.*

(a) *The curvature of M is strictly negative.*

(b) *The Euler characteristic of M is nonzero.*

(c) *The center of $\pi_1(M)$ is trivial.*

(d) *The first Betti number of M is zero.*

Then the only isometry of M homotopic to the identity is the identity itself. In particular, the group of isometries is finite.

Corollary 4. *Let M be as in Theorem* 4, *and let $I(M)$ denote the group of isometries of M. Then*

(a) *$I(M)^0 = T^k$ where k is the rank of the center of $\pi_1(M)$.*

(b) *If $f \in I(M) \sim I(M)^0$, then $f \not\cong \mathbf{1}$.*

Note. The action of $I(M)^0$ is locally free and, in particular, fixed-point free. However, in general it may not be free. Consider, for example, the standard flat Klein bottle $K = \boldsymbol{R}^2/\pi$ where π is the group of euclidean motions generated by $A, B: \boldsymbol{R}^2 \to \boldsymbol{R}^2$ where $A(x, y) = (x, y + 1)$ and $B(x, y) = (x + 1, -y)$. The center of π is the infinite cyclic group generated by B^2.

$I(K)^0 \simeq S^1$ and corresponds to the group of translations of $\boldsymbol{R}^2$ in the x-direction. The orbits of $I(K)^0$ corresponding to the lines $y = n/2$, $n = 0, \pm 1, \pm 2, \cdots$, are singular and have $1/2$ the length of the other orbits on K. We note that the orbit structure in this case is exemplary of the general one since the isotropy subgroup of $I(M)^0$ for any singular orbit must be finite.

7. Manifolds of finite volume

Several of the theorems discussed above can be shown to hold in the following somewhat more general context.

Theorem 5. *Let M be a complete riemannian manifold of nonpositive curvature and finite volume. Then the center theorem and Proposition* 2 *hold for M. If, furthermore, $\pi_1(M)$ is finitely generated, then Theorems* 3, 4 *and all their corollaries hold for M.*

Proof. Let $\pi: \tilde{M} \to M$ be the universal covering of M, and $\Pi \simeq \pi_1(M)$ be the group of deck transformations of the covering. Let Z be a nontrivial element of the center of Π, and consider the function $\phi: \tilde{M} \to \boldsymbol{R}$ given by $\phi(p) = d^2(p, Z(p))$. Then ϕ is a convex function [2, § 4.2] which, since Z centralizes Π, projects to a convex function on M and thus, by [2, § 2.2] is constant. Hence, if γ is any infinite geodesic in $\tilde{M}$, then γ and $Z\gamma$ are of the same type. It follows that the fundamental vector field v_z on $\tilde{M}$ (cf. § 1) is globally parallel. The remainder of the proof of the center theorem can now be carried out exactly as in [12].

Observe that by using the fact that any convex function on M is a constant [2, § 2.2] one can prove Proposition 2 for M in exactly the same manner as above.

It remains only to establish Theorem 3. Let $v_1, \cdots, v_k$ be a maximal set of linearly independent globally parallel vector fields on M. As in § 5, we get a splitting $\tilde{M} = \boldsymbol{R}^k \times \tilde{M}'$ where each $G \in \Pi$ satisfies equation (5.1), and if G lies in the center $\mathscr{Z}$ of Π, then $G(x, y) = (x + c_G, y)$. Let $\mathscr{N} = \{G \in \Pi : c_G = 0\}$. Then $\mathscr{N}$ is a normal subgroup which contains $[\Pi, \Pi]$. Dividing out by the group $\mathscr{Z} \times \mathscr{N} \subset \pi$, we obtain a covering

$$T^p \times \boldsymbol{R}^q \times M' \to M ,$$

where T^p is a flat torus, $p = \operatorname{rank}(\mathscr{Z})$, $p + q = k$, and each element A of the abelian deck group $\mathscr{A} \simeq \Pi/(\mathscr{Z} \times \mathscr{N})$ can be written as

$$A(\tau, x, y) = (\tau + \tau_A, x + x_A, a(y)) \tag{7.1}$$

for all $(\tau, x, y) \in T^p \times \boldsymbol{R}^q \times M'$.

We shall prove that $q = 0$ (and thus establish the theorem). Observe that since Π is finitely generated, Lemma 3 is valid here. It follows that if for any $A \in \mathscr{A}$ there is a $y \in M'$ such that $a(y) = y$ (cf. (7.1)), then there is an integer

$n \geq 1$ such that $A^n = 1$. Let $T = \{A \in \mathscr{A} : A^n = 1 \text{ for some } n \geq 1\}$. Then, since $\mathscr{A}$ is finitely generated and abelian, T is a finite group of order, say, m. Hence the group $\mathscr{A}^m = \{A^m : A \in \mathscr{A}\}$ acts *freely* on the factor M' (by $y \mapsto a(y)$). We assert that this action is also properly discontinuous. If it were not, there would exist $y_0 \in M'$ and a sequence $\{A_n\}_{n=1}^\infty$ in $\mathscr{A}^m$ such that $a_n(y_0) \to y_0$. By passing to a subsequence we may assume that there is an isometry a_∞ of M' such that $a_n \to a_\infty$ in the compact-open topology [7, Th. 4.7]. Clearly, for all $A \in \mathscr{A}$ we have that $a_\infty a = a a_\infty$. We now define a sequence of isometries $\{G_n\}$ of $T^p \times \boldsymbol{R}^q \times M'$ by setting

$$G_n(\gamma, x, y) = (\gamma, x, a_\infty^{-1} a_n(y))$$

for all $(\gamma, x, y) \in T^p \times \boldsymbol{R}^q \times M'$. Then $G_n \to 1$ in the group of isometries of $T^p \times \boldsymbol{R}^q \times M'$. Since each G_n commutes with all of $\mathscr{A}$, we have that $\{G_n\}$ projects to a sequence $\{\bar{G}_n\}$ of isometries of M which converge to the identity in the group $I(M)$ of isometries of M. Thus, for some sufficiently large n_0, we have $\bar{G}_{n_0} \in I(M)^0$ (and therefore $\bar{G}_{n_0} \simeq \mathbf{1}$). It now follows from Proposition 2 that $\bar{G}_{n_0}$ defines a globally parallel vector field v on M which, by the construction, is independent of $v_1, \cdots, v_k$ if it is nonzero. However, since a_{n_0} acts freely on M' and $a_\infty(y_0) = y_0$, G_{n_0} is nontrivial and thus v *is* nonzero. This contradicts the maximality of $v_1, \cdots, v_k$ and establishes the assertion.

We now show that $q = 0$. Observe that since the index of $\mathscr{A}^m$ in $\mathscr{A}$ is finite, the manifold $M_1 = (T^p \times \boldsymbol{R}^q \times M')/\mathscr{A}^m$ has finite volume. However, since $\mathscr{A}^m$ acts properly discontinuously on M', we can find a set of the type $T^p \times \boldsymbol{R}^q \times U$, where U is an open subset of M', contained in a fundamental domain for the action of $\mathscr{A}^m$ on $T^p \times \boldsymbol{R}^q \times M'$. It follows that q must be zero, and the theorem is proved.

Corollary 5. *Let M be a complete nonpositively curved manifold of finite volume and negative definite Ricci curvature. Then $I(M)^0 = \{e\}$.*

Proof. Suppose there were an isometry $f \in I(M)^0 \sim \{e\}$. Then by Proposition 2 there would exist a nonzero globally parallel vector field v on M. Clearly, for any tangent vector field X on M we have

$$\langle R_{v,X} v, X \rangle = \langle (\nabla_v \nabla_X v - \nabla_X \nabla_v v - \nabla_{[v,X]} v, X) \rangle \equiv 0 \, .$$

Thus the Ricci curvature Ric $(v, v) \equiv 0$, contrary to assumption.

Finally we mention a corollary related to Gottlieb's theorem.

Corollary 6. *Let π be a finitely generated group such that $K(\pi, 1)$ has the homotopy type of a complete nonpositively curved riemannian manifold of finite volume. Let $\mathscr{Z}$ be the center of π and assume $\mathscr{Z}$ has rank k. Then there exists a subgroup π' of finite index in π with $\mathscr{Z} \subset \pi'$ such that the fibration*

$$K(\mathscr{Z}, 1) \to K(\pi', 1) \to K(\pi'/\mathscr{Z}, 1)$$

can be realized as a (finite demensional) differentiable fibre bundle.

Proof. Let M be the complete nonpositively curved manifold with finite volume and fundamental group π. As in the proof of Theorem 5 there is a covering $T^k \times M' \to M$ whose deck group $\mathscr{A}$ is finitely generated and abelian. Furthermore, there is an integer $m > 0$ such that $\mathscr{A}^m$ acts freely and properly discontinuously on the second factor. Let $\overline{M} = (T^k \times M')/\mathscr{A}^m$ and set $\pi' = \pi_1(\overline{M})$. Then π' has finite index in π, $\overline{M} = K(\pi', 1)$, and we clearly have a differentiable (in fact, principal) fibre bundle $T^k \to \overline{M} \to \overline{\overline{M}}$ where $\overline{\overline{M}}$ is diffeomorphic to $M'/\mathscr{A}^m$. The manifold $\overline{\overline{M}}$ inherits a natural nonpositively curved complete Riemannian metric from the factor M'. Thus $\overline{\overline{M}} = K(\pi_1(\overline{\overline{M}}), 1) = K(\pi'/\mathscr{Z}, 1)$ and the theorem is proved.

Remarks 1. Let π be as in Corollary 6. From Corollary 6 and a spectral sequence argument we can conclude Gottlieb's theorem: *If* $\chi(\pi)$ $(= \chi(K(\pi, 1)))$ $\neq 0$, *then* $\mathscr{Z} = \{1\}$ (see [11, § 4.3]) provided $H_*(K(\pi, 1), \boldsymbol{R})$ is finite dimensional. As noted above, if $K(\pi, 1)$ is a *compact* nonpositively curved manifold, the theorem is easy.

2. It can be shown by example that finite volume does not imply that π is finitely generated. It would be interesting to know whether Corollary 6 continues to hold if the condition that π be finitely generated is dropped.

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University of California, Berkeley